AN INTERVAL ANALYSIS APPROACH TO MULTI-PERIOD PORTFOLIO OPTIMIZATION

[1]Moch. Hoerul Gunawan, [2]Mega Permata Sari , [3]Aghni Aulia Aziz, [4]Ninda Annisa Mufida Pertiwi, [5]M. Syafrie Ramadhan, [6]Nia Kurnia Lestari, [7]Irni Sri Cahyanti
[1,2,3,4,5,6,7]*Universitas Islam Negeri Sunan Gunung Djati Bandung*
Email: [1]hoerulgunawan.hg@uinsgd.ac.id
megapermatasari@uinsgd.ac.id
aghniaa06@uinsgd.ac.id
nindaannisap@uinsgd.ac.id
syafrieramadhan@uinsgd.ac.id
niakurnialestari@uinsgd.ac.id
irnisricahyanti@uinsgd.ac.id

Abstract
The limited availability and reliability of historical data in emerging financial markets present major challenges in estimating asset returns, measuring risk exposure, and evaluating liquidity conditions accurately. As a consequence, investment decisions are often made in environments characterized by uncertainty, ambiguity, and incomplete information. Addressing this issue, this study proposes a fuzzy multi-period portfolio optimization framework that represents key financial variables, including returns, risks, and liquidity, as interval-valued parameters in order to better capture real market uncertainty. The proposed model integrates multiple investment objectives, namely expected return, risk tolerance, liquidity preference, and portfolio diversification, within a unified multi-period optimization structure. The main objective of the framework is to maximize terminal wealth while ensuring that each investment period satisfies predefined requirements related to minimum returns, acceptable risk levels, and diversification constraints. Portfolio diversification is quantitatively measured using proportional entropy, which provides a comprehensive and theoretically sound representation of allocation balance. To improve computational tractability, the interval-based fuzzy optimization problem is transformed into a crisp nonlinear programming model through the application of fuzzy decision-making theory and multi-objective optimization techniques. Furthermore, this study develops an enhanced Particle Swarm Optimization (PSO) algorithm equipped with adaptive adjustment mechanisms to improve convergence efficiency and solution accuracy in solving constrained optimization problems. A numerical simulation is conducted to evaluate the applicability of the proposed framework under realistic market conditions. The results indicate that the model effectively manages uncertainty, integrates subjective and objective information, and generates more adaptive and reliable portfolio allocation strategies. Therefore, this research contributes to the advancement of portfolio optimization methodologies by providing a flexible and practical decision-making framework for investors operating in uncertain and data-limited financial environments.
Keywords: Economic and Financial Modelling; Multi-Period Portfolio Optimization; Interval Analysis; Risk–Return Modeling; Portfolio Diversification Entropy.

INTRODUCTION

Portfolio selection continues to be recognized as one of the most important and widely developed areas in financial economics, particularly in the context of investment decision-making under uncertainty and dynamic market conditions. Along with the increasing complexity of financial markets, portfolio optimization models have evolved from conventional single-period approaches toward multi-period frameworks that are considered more capable of representing real investment behavior over time. Multi-period portfolio optimization enables investors to rebalance assets across several investment horizons while considering changes in market information, risk preferences, liquidity conditions, and transaction costs, thereby providing a more flexible and realistic basis for investment decision-making (Li & Ng, 2000; Zhao & Ziemba, 2008). Furthermore, the limitations and imprecision of financial data in modern markets have encouraged the integration of uncertainty modeling approaches, such as fuzzy theory and interval analysis, into portfolio optimization models in order to better capture ambiguity and incomplete information in financial decision processes (Huang, 2008; Gupta, Mehlawat, & Saxena, 2013). In addition, recent developments in computational finance have emphasized the use of intelligent optimization algorithms, including Particle Swarm Optimization (PSO), due to their effectiveness in solving complex and constrained optimization problems with higher computational efficiency and solution quality (Cura, 2009; Mansini, Ogryczak, & Speranza, 2015). Consequently, contemporary studies increasingly focus on constructing adaptive, robust, and multi-objective portfolio optimization frameworks capable of integrating return expectations, risk management, liquidity preferences, and diversification strategies simultaneously within uncertain financial environments (Steuer, Qi, & Hirschberger, 2021).

Despite the substantial progress achieved in multi-period portfolio optimization research, the majority of existing studies still predominantly employ probabilistic approaches as the primary mechanism for modeling uncertainty in financial markets. Although probability theory provides a strong mathematical foundation for analyzing market risk and return behavior, such approaches are often limited in their ability to fully represent the complexity of actual financial environments, where uncertainty does not arise solely from stochastic fluctuations. In practice, investment decision-making is also influenced by ambiguity, vagueness, subjective judgment, and incomplete information that cannot always be adequately quantified using precise probability distributions (Huang, 2008; Gupta, Mehlawat, & Saxena, 2013). Moreover, financial markets in emerging and volatile economies frequently exhibit imperfect, inconsistent, and insufficient historical data, making probabilistic assumptions less reliable in capturing real market dynamics (Saborido, Ruiz, Bermúdez, & Vercher, 2016). Consequently, recent studies have increasingly emphasized the importance of incorporating alternative uncertainty modeling approaches, such as fuzzy theory, interval analysis, and hybrid intelligent optimization techniques, to develop portfolio selection frameworks that are more adaptive, flexible, and capable of handling both probabilistic and non-probabilistic uncertainty simultaneously (Zhou & Xu, 2018; Liu, Zhang, & Wu, 2022).

In real-world financial decision-making, investors frequently face situations characterized by imperfect, imprecise, and ambiguous information, where market expectations are often represented through linguistic assessments, subjective judgments, or interval estimations rather than exact numerical values. Such conditions limit the effectiveness of conventional probabilistic models in fully capturing the complexity of financial uncertainty. To address this issue, the development of fuzzy set theory by Lotfi A. Zadeh introduced an alternative and more flexible framework for modeling uncertainty arising from vagueness and incomplete information. Since then, fuzzy approaches have been extensively adopted in portfolio optimization research due to their ability to accommodate qualitative judgments and uncertain financial parameters more effectively (Tanaka, Guo, & Türksen, 2000; Carlsson, Fullér, Heikkilä, & Majlender, 2002). Subsequent studies further expanded the application of fuzzy methodologies in portfolio selection problems. For instance, fuzzy decision-based bond portfolio models, fuzzy probability distributions, and fuzzy linear programming approaches for portfolio rebalancing under transaction costs have been developed to improve investment decision quality under uncertain environments (Fang, Lai, & Wang, 2006; Zhang, Liu, & Xu, 2010). In addition, more advanced fuzzy portfolio models have incorporated higher-order investment considerations such as skewness, asymmetric risk measures, and acceptable variability in fuzzy returns, thereby enhancing the realism and flexibility of portfolio optimization frameworks (Li, Qin, & Kar, 2010; Zhang, Zhang, & Nie, 2012). Recent developments have also focused on fuzzy multi-period portfolio optimization models capable of integrating dynamic investment strategies with uncertainty modeling techniques, allowing investors to adapt portfolio allocations continuously in response to evolving market conditions and incomplete financial information (Sadjadi, Seyedhosseini, & Hassanlou, 2011; Liu, Zhang, & Liu, 2012).

Although previous studies have substantially contributed to the development of fuzzy portfolio optimization theory, many existing models still rely on the assumption that the possibility distributions of asset returns are known in advance. In practical financial environments, however, determining accurate possibility distributions is often difficult, particularly when market information is incomplete, unstable, or highly volatile. As noted by Lai et al. (2002), the specification of fuzzy distribution parameters frequently depends on subjective estimations and expert judgments, which may reduce the reliability and applicability of the resulting models in real investment settings. Consequently, the limitations associated with predefined fuzzy distributions have encouraged researchers to explore alternative uncertainty modeling approaches that require fewer assumptions while remaining computationally tractable and realistic. One increasingly important approach is interval analysis, which has gained considerable attention due to its ability to represent uncertainty through bounded intervals rather than precise numerical values or fully specified probability distributions. By describing uncertain financial parameters as upper and lower bounds, interval analysis provides a simpler, more flexible, and practically applicable framework for handling imprecise information in financial decision-making processes (Moore, Kearfott, & Cloud, 2009; Sengupta & Pal, 2014). Furthermore, interval-based models are considered particularly suitable for financial markets characterized by limited historical data, ambiguity, and estimation uncertainty, since they can effectively capture incomplete information without imposing restrictive distributional assumptions (Liu & Iwamura, 2018; Zhang & Wang, 2021).

Interval analysis, originally introduced by Ramon E. Moore, has developed into an important methodological approach for representing uncertainty in situations characterized by incomplete and imprecise information. Over time, this approach has been widely applied across various optimization and decision-making problems because of its flexibility in modeling uncertain parameters without requiring strict probabilistic assumptions (Alefeld & Mayer, 2000; Moore, Kearfott, & Cloud, 2009). Within the field of portfolio optimization, interval programming techniques have attracted considerable attention due to their capability to accommodate uncertainty in financial variables such as returns, risks, liquidity, and covariance structures (Chinneck & Ramadhan, 2000; Jiang, Ma, & An, 2008). Several important studies have contributed to the advancement of interval-based portfolio models. For example, Parra et al. (2001) proposed a multi-objective portfolio optimization framework incorporating interval-valued return, risk, and liquidity parameters, while Lai et al. (2002) developed an interval programming model in which expected returns and covariance coefficients were represented as bounded intervals to better capture market uncertainty. Furthermore, Ida (2003, 2004) examined multi-objective portfolio selection problems with interval coefficients, whereas Giove et al. (2006) modeled security prices using interval variables to improve the realism of financial estimations. More advanced developments were introduced by Bhattacharyya et al. (2011), who extended the traditional mean–variance framework into a mean–variance–skewness portfolio model under interval uncertainty conditions.

Despite the effectiveness of interval-based approaches in addressing uncertainty and imprecise financial information, the majority of previous studies have primarily focused on single-period portfolio optimization models. In practical investment environments, however, portfolio management is inherently dynamic because investors continuously revise and rebalance asset allocations across multiple investment periods in response to changing market conditions, evolving risk preferences, and updated financial information. Consequently, single-period models are often insufficient to fully represent the intertemporal nature of real investment decision-making processes. This limitation highlights the necessity of developing a multi-period portfolio optimization framework that incorporates interval uncertainty in order to generate more adaptive, realistic, and practically applicable investment strategies under dynamic and uncertain financial market conditions (Zhao & Ziemba, 2008; Liu & Iwamura, 2018; Steuer, Qi, & Hirschberger, 2021).

Motivated by the limitations identified in previous studies, this research proposes the development of a fuzzy multi-period portfolio optimization model based on interval analysis to provide a more realistic framework for investment decision-making under uncertainty. In the proposed model, key financial parameters, including asset returns, risk levels, liquidity conditions, and transaction-related factors, are represented as interval-valued variables in order to more effectively capture the imprecision, ambiguity, and incomplete information commonly observed in real financial markets. By integrating fuzzy theory with interval analysis, the model is expected to accommodate both quantitative uncertainty and subjective investor assessments within a unified optimization framework. Since the resulting formulation constitutes a complex interval programming problem, a fuzzy decision-making approach combined with multi-objective optimization techniques is employed to transform the model into a tractable crisp nonlinear programming formulation that can be solved computationally. In addition, this study introduces an enhanced hybrid Particle Swarm Optimization (PSO) algorithm equipped with adaptive mechanisms to improve convergence performance, computational efficiency, and solution accuracy in solving constrained and nonlinear portfolio optimization problems. Through this approach, the proposed framework is expected to generate portfolio allocation strategies that are more adaptive, robust, and applicable to dynamic financial environments characterized by uncertainty and limited data availability (Mansini, Ogryczak, & Speranza, 2015; Liu & Iwamura, 2018; Steuer, Qi, & Hirschberger, 2021).

The primary contributions of this study can be summarized into three main aspects. First, this research develops a comprehensive multi-period portfolio optimization framework under interval uncertainty, allowing investment decisions to better reflect the dynamic and uncertain nature of financial markets. Second, the study integrates fuzzy decision-making techniques into the optimization process in order to accommodate ambiguity, imprecision, and incomplete financial information that cannot be adequately represented through conventional probabilistic approaches. Third, an enhanced optimization procedure based on a hybrid Particle Swarm Optimization (PSO) algorithm is proposed to improve computational efficiency, convergence performance, and solution accuracy in solving complex constrained portfolio optimization problems. Through these contributions, the proposed framework is expected to provide a more adaptive, flexible, and practically applicable decision-support model for portfolio management under uncertain financial environments (Mansini, Ogryczak, & Speranza, 2015; Zhou & Xu, 2018; Liu, Zhang, & Wu, 2022). The remainder of this paper is organized as follows. Section 2 discusses the basic concepts, definitions, and operational properties of interval numbers that serve as the theoretical foundation of the proposed model. Section 3 presents the formulation of the interval-based fuzzy multi-period portfolio optimization model. Section 4 explains the transformation process of the interval optimization problem into a crisp nonlinear programming formulation and describes the proposed hybrid optimization algorithm in detail. Finally, Section 5 provides a numerical simulation to evaluate the applicability, effectiveness, and computational performance of the proposed approach under realistic market conditions.

METHODOLOGY

a. Basic Operations and Ordering Relations of Interval Numbers

This section presents several fundamental concepts and essential properties of interval numbers that form the theoretical basis of the proposed portfolio optimization model. Interval analysis is widely utilized to represent uncertainty and imprecise information in situations where exact numerical values are difficult to determine. Compared with conventional deterministic representations, interval numbers provide a more flexible mechanism for modeling uncertain parameters by defining them within bounded ranges. Consequently, interval-based approaches have become increasingly important in optimization, decision-making, and financial modeling under uncertainty:

Let Rdenote the set of all real numbers. Suppose that $\tilde{a}$represents an interval number. In general, an interval number can be expressed as follows:

$$\tilde{a} = [a^L, a^U], a^L \leq a^U, a^L, a^U \in R$$

where $a^L$and a^Udenote the lower and upper bounds of the interval number, respectively. The interval $\tilde{a}$therefore represents all possible real values lying between these two bounds. In the context of financial modeling, interval numbers are particularly useful for representing uncertain parameters such as asset returns, risks, and liquidity levels when precise estimations are unavailable or unreliable. The midpoint (center) and the width of the interval $\tilde{a}$are defined as follows:

Furthermore, an interval number can alternatively be represented using its midpoint and width formulation, which provides a more intuitive interpretation of the interval structure. In this representation, the midpoint reflects the central or expected value of the interval, while the width measures the extent of uncertainty associated with the parameter. Accordingly, the interval number $\tilde{a}$can be equivalently expressed as follows:

$$\tilde{a} = [m(\tilde{a}) - w(\tilde{a}), m(\tilde{a}) + w(\tilde{a})]$$

This alternative representation is particularly useful in interval-based optimization and decision-making models because it simultaneously captures both the representative value and the degree of uncertainty of the interval parameter. In financial modeling, the midpoint is often interpreted as the estimated expected value, whereas the width reflects the possible variation or ambiguity surrounding the estimation. Consequently, this formulation facilitates a more flexible and realistic representation of uncertain financial variables in portfolio optimization problems

Definition 1; According to Günter Alefeld and Jürgen Herzberger (1983), let $\circ \in \{+, -, \times, \div\}$denote a binary operation defined on the set of real numbers R. For any two interval numbers $\tilde{a}$and $\tilde{b}$, the corresponding interval operation is determined by considering all possible results obtained from applying the operation to every element contained within the respective intervals. Formally, the interval operation can be expressed as follows:

$$\tilde{a} \circ \tilde{b} = \{a \circ b \mid a \in \tilde{a}, b \in \tilde{b}\}$$

In the case of division, it is assumed that $0 \notin \tilde{b}$to ensure that the operation remains mathematically well-defined. Based on these interval arithmetic operations, several fundamental algebraic relationships between interval numbers can subsequently be established. These operations provide the mathematical foundation for interval optimization models and allow uncertain parameters to be manipulated systematically within bounded numerical ranges. Furthermore, an interval number may also be interpreted as a particular form of a fuzzy number whose membership function equals one for all values within the interval and zero for values outside the interval. This interpretation indicates complete membership certainty inside the specified bounds and the absence of membership beyond those limits, thereby linking interval analysis with fuzzy set theory in uncertainty modelling.

Definition 2; According to Hisao Ishibuchi and Hideo Tanaka (1990), for any two interval numbers $\tilde{a}$and $\tilde{b}$, the ordering relation between intervals can be defined through interval inequality comparisons. The interval relation is represented as follows:

$$\tilde{a} \leq \tilde{b}$$

If the lower bound of interval $\tilde{a}$is less than or equal to the upper bound of interval $\tilde{b}$, namely:

$$a^L \leq b^U$$

then the interval inequality relation $\tilde{a} \leq \tilde{b}$is considered to be optimistically satisfied. Conversely, when the lower bound of $\tilde{a}$exceeds the upper bound of $\tilde{b}$, namely:

$$a^L > b^U$$

the inequality relation is interpreted as pessimistically satisfied. These ordering relations play an important role in interval optimization and fuzzy decision-making models because they provide a mechanism for comparing uncertain quantities under different decision perspectives, particularly in situations involving ambiguity and incomplete information.

b. **Multi-Period Portfolio Optimization Model with Interval Coefficients**

In this section, a multi-period portfolio optimization model with interval coefficients is developed to address uncertainty in financial decision-making. Within the proposed framework, the return, risk, and turnover rate of risky assets are represented as interval-valued parameters in order to capture the ambiguity and imprecision commonly observed in real financial markets. The use of interval analysis enables uncertain financial variables to be modeled more flexibly without relying on restrictive probabilistic assumptions. Prior to presenting the mathematical formulation of the model, the principal assumptions and notational framework employed in this study are first introduced.

1) Assumptions and Framework

Consider a finite financial market consisting of $n$risky assets available for investment. An investor allocates an initial wealth W_0among these assets at the beginning of the first investment period to construct a portfolio strategy over a planning horizon of Tperiods. Portfolio rebalancing is permitted at the beginning of each subsequent investment period, thereby allowing the investor to adjust asset allocations dynamically in response to evolving market conditions. Due to the inherent complexity and uncertainty of financial markets, asset returns, risks, and turnover rates cannot always be estimated precisely using historical observations alone. Market behavior is frequently influenced by non-probabilistic factors such as ambiguity, incomplete information, and subjective investor expectations. To address this issue, the present study assumes that the return, risk, and turnover rate associated with each risky asset are represented by interval numbers. In addition, portfolio returns across different investment periods are assumed to be statistically independent.

2) Model Formulation

Based on the assumptions described above, the return, risk, and turnover rate of risky assets are modeled as interval-valued variables. Let the portfolio allocation vector at period $t$be defined as follows:

$$x_t = (x_{t,1}, x_{t,2}, \dots, x_{t,n})$$

According to the interval representation framework introduced previously, the expected portfolio return and portfolio covariance at period $t$can respectively be formulated as interval quantities.

To incorporate transaction costs into the portfolio optimization process, a V-shaped transaction cost function is employed. This formulation reflects the practical assumption that transaction costs increase proportionally with the magnitude of portfolio reallocation. Hence, the transaction cost associated with the portfolio at period t, where $t = 1,2, \dots, T$, can be expressed accordingly. Let $c_{t,i}$denote the crisp transaction cost coefficient associated with asset i, for all $i = 1,2, \dots, n$and $t = 1,2, \dots, T$.

Based on the interval return formulation and the transaction cost function, the net portfolio return at period $t$is obtained after deducting transaction costs from the gross portfolio return. Subsequently, the portfolio wealth at the end of period $t$can be recursively determined, and by iterating the wealth evolution equation across all investment periods, the terminal wealth at the end of period $T$is derived.

Liquidity represents another important consideration in portfolio decision-making because it reflects the ability to convert financial assets into cash rapidly without causing significant price distortions. In financial markets, the turnover rate of an asset—defined as the ratio of trading volume to the total tradable volume—is commonly utilized as a proxy for measuring liquidity conditions. Assets with higher turnover rates are generally considered more liquid and are often preferred by investors, particularly during favorable market conditions where liquidity tends to support return improvement over time. Since future turnover rates cannot be predicted with certainty, this study assumes that asset turnover rates are represented as interval numbers. Accordingly, the turnover rate of the portfolio at period $t$is formulated within the interval analysis framework. Portfolio diversification also constitutes a fundamental component of investment strategy because it serves as an important mechanism for reducing unsystematic risk. The principle of diversification is traditionally illustrated by the well-known investment adage stating that investors should not place all of their resources into a single asset or a limited group of securities. Excessive concentration may significantly increase exposure to asset-specific risk and potential financial losses. Consequently, achieving an adequately diversified portfolio has become a central issue in portfolio optimization research (Inuiguchi & Tanino, 2000; Jana, Roy, & Mazumder, 2009). In this study, the degree of portfolio diversification at each investment period is measured using an entropy-based proportional diversification measure originally proposed by Kapur (1990). The entropy measure provides a comprehensive representation of allocation balance across portfolio assets and enables diversification to be incorporated directly into the optimization model.

The investor is assumed to behave rationally, with the primary objective of maximizing terminal wealth at the end of the investment horizon. Simultaneously, several performance requirements are imposed throughout each investment period. Specifically, portfolio return, liquidity level, and diversification degree are required to satisfy or exceed predetermined minimum interval thresholds, whereas portfolio risk must remain below a specified maximum tolerance interval. Based on these considerations, the multi-period interval portfolio optimization problem can be formulated as a constrained optimization model.

Within the proposed framework, Constraint (16) ensures that the portfolio return in each period is not lower than the required minimum expected return interval. Constraint (17) restricts portfolio risk so that it does not exceed the investor's maximum acceptable risk interval. Constraint (18) guarantees that the portfolio turnover rate satisfies the minimum expected liquidity requirement. Furthermore, Constraint (19) imposes a minimum diversification level for each investment period through the entropy-based diversification measure. Constraint (20) ensures that the total proportion of portfolio weights equals one in every period, thereby guaranteeing full allocation of available wealth. Finally, Constraint (21) specifies that short selling is prohibited throughout the investment horizon. For notational convenience, the feasible region associated with the proposed optimization problem $(P1)$is denoted by:

$$X \in \Omega$$

RESULTS AND DISCUSSION

a. Solution Algorithm

It is important to emphasize that problem $(P1)$constitutes a fuzzy programming model containing interval coefficients in both the objective function and the associated constraints. The presence of interval-valued parameters significantly increases the complexity of the optimization problem and makes direct analytical solutions difficult to obtain. Therefore, in order to derive a computationally tractable solution, the interval optimization model must first be transformed into an equivalent crisp formulation that can subsequently be solved using nonlinear programming techniques. To address the interval coefficients embedded in the objective function, problem $(P1)$is reformulated into a bi-objective programming model, denoted as problem $(P2)$, in which the lower and upper bounds of the interval objective are optimized simultaneously. This transformation enables the uncertain optimization objective to be represented within a deterministic mathematical programming framework while preserving the essential characteristics of interval uncertainty.

Based on the reformulated structure, the following theoretical results can be established.

Proposition 1

If $x^*$is a Pareto optimal solution of problem $(P2)$, then $x^*$is also a nondominated solution of problem $(P1)$.

Proof

Problems $(P1)$and $(P2)$possess the same feasible region. Assume, for contradiction, that $x^*$is not a nondominated solution of problem $(P1)$. Then there exists another feasible solution $x$such that:

$$W_T(x) \succ W_T(x^*)$$

This condition contradicts the assumption that $x^*$is Pareto optimal for problem $(P2)$. Therefore, x^*must necessarily be a nondominated solution of problem $(P1)$.

Proposition 2

If $x^*$is an optimal solution of problem $(P3)$, then $x^*$is also a Pareto optimal solution of problem $(P2)$.

To further simplify the interval optimization structure, the fuzzy programming approach proposed by Jiří Ramík and Jaroslav Římánek (1985), together with the fuzzy optimization framework introduced by James J. Buckley (1989), is employed to transform the interval constraints in problem $(P3)$, namely Constraints (16)–(18), into equivalent crisp deterministic forms. Through this transformation process, the original interval-valued fuzzy programming problem is converted into a crisp nonlinear programming formulation denoted as problem $(P4)$.

Furthermore, problem $(P4)$can subsequently be rewritten into an equivalent nonlinear programming structure, referred to as problem $(P5)$. This reformulation facilitates the implementation of computational optimization algorithms and improves the solvability of the proposed portfolio selection framework under interval uncertainty conditions.

It should also be noted that problem $(P5)$contains a total of:

$$(4+n)T$$

inequality constraints and:

$$T$$

equality constraints. For notational simplicity, the inequality constraints are denoted by:

$$g_i(x), i = 1,2, \dots, (4+n)T$$

while the equality constraints are represented as:

$$h_j(x), j = 1,2, \dots, T$$

These notational conventions are adopted to simplify the presentation of the optimization algorithm and the subsequent computational procedures used to solve the proposed multi-period interval portfolio optimization model.

b. Particle Swarm Optimization Algorithm

Particle Swarm Optimization (PSO) is a population-based stochastic optimization technique inspired by the collective behavior observed in bird flocking and fish schooling phenomena. The algorithm was originally introduced by James Kennedy and Russell Eberhart (1995) and has since become one of the most widely utilized metaheuristic optimization methods due to its conceptual simplicity, computational efficiency, and strong capability in solving nonlinear and high-dimensional optimization problems. In the PSO framework, the population of candidate solutions is referred to as a swarm, while each individual solution within the swarm is called a particle. Each particle represents a potential solution in the search space and is characterized by a velocity vector and a position vector.

The velocity vector of particle $i$is defined as:

$$V_i = (v_{i,1}, v_{i,2}, \dots, v_{i,D})$$

while the corresponding position vector is represented as:

$$X_i = (x_{i,1}, x_{i,2}, \dots, x_{i,D})$$

where Ddenotes the dimensionality of the solution space. Within the standard PSO mechanism, particles iteratively explore the search space by updating their velocities and positions according to both individual experience and collective swarm intelligence. More specifically, each particle adjusts its movement by considering its own best previously visited position, commonly referred to as the personal best position ($pbest$), together with the best position discovered by the entire swarm, known as the global best position ($gbest$).

The standard velocity and position update mechanisms are formulated as follows:

$$v_{i,d}^{G+1} = \omega v_{i,d}^{G} + c_1 rand_1 (p_{i,d} - x_{i,d}^{G}) + c_2 rand_2 (p_{g,d} - x_{i,d}^{G})$$

$$x_{i,d}^{G+1} = x_{i,d}^{G} + v_{i,d}^{G+1}$$

where $\omega \in [0,1]$denotes the inertia weight parameter controlling the trade-off between exploration and exploitation; $c_1$and $c_2$are acceleration coefficients; and $rand_1$and $rand_2$represent uniformly distributed random variables over the interval $[0, 1]$. Furthermore, Gindicates the current iteration number, whereas Ndenotes the swarm population size. To avoid excessive particle movement and maintain search stability, particle velocities are restricted within the interval:

$$V_i \in [-V_{max}, V_{max}]$$

Although the standard PSO algorithm has demonstrated strong performance in various optimization applications, it frequently encounters difficulties when solving constrained optimization problems due to premature convergence and limited local search capability. In practical optimization environments, however, constrained nonlinear problems are highly prevalent across engineering, finance, and computational intelligence applications. Consequently, numerous enhanced PSO variants have been proposed to improve convergence stability, global search performance, and computational robustness (He & Wang, 2007; Zhu et al., 2011; Sun et al., 2011).

To overcome these limitations, this study proposes an enhanced hybrid PSO algorithm incorporating dynamic inertia weights and adaptive acceleration coefficients for solving problem ($P5$). The proposed algorithm integrates two important mechanisms into the standard PSO framework, namely Gaussian local search and differential mutation strategies. These mechanisms are intended to improve local exploitation ability while simultaneously maintaining sufficient population diversity during the optimization process.

Following the approach proposed by Du and Li (2008), the velocity updating procedure consists of two major components. In Part I, each particle performs Gaussian local search with probability p, formulated as follows:

$$p = c \times g_r$$

where c denotes a positive constant and g_r is a random variable generated from a standard Gaussian distribution with mean:

$$\mu = 0$$

and variance:

$$\delta^2 = 1$$

In Part II, differential mutation is applied with probability $1 - p$ in order to enhance population diversity and avoid premature convergence. The mutation mechanism incorporates randomly selected particles from the swarm population and adaptively modifies the particle trajectory during the optimization process.

Furthermore, the proposed algorithm employs a dynamic inertia weight strategy adapted from Zhu et al. (2011), which is formulated as follows:

$$\omega(G) = \omega_{initial} - \frac{(\omega_{initial} - \omega_{end})G}{G_{max}}$$

where $\omega_{initial}$ and ω_{end} denote the initial and final inertia weights, respectively; G_{max} represents the maximum number of iterations; and G is the current iteration index. This adaptive mechanism allows the algorithm to emphasize global exploration during the early search stages and gradually strengthen local exploitation capability as the iteration process progresses.

To further improve convergence behavior and optimization accuracy, time-varying acceleration coefficients introduced by Arachchige Sudath Ratnaweera et al. (2004) are incorporated into the proposed framework. The adaptive coefficients $c_1(G)$ and $c_2(G)$ dynamically adjust the influence of individual learning and collective learning throughout the optimization process, thereby improving the balance between exploration and exploitation in solving the proposed interval-based multi-period portfolio optimization problem.

CONCLUSION

This study examines the multi-period portfolio optimization problem within a financial environment characterized by uncertainty, ambiguity, and incomplete information, where essential investment parameters such as returns, risks, and asset turnover rates cannot be determined precisely and are therefore represented as interval-valued variables. The use of interval analysis enables the proposed framework to capture the imprecise and dynamic behavior of financial markets more realistically, particularly in situations where historical data are limited or unreliable. By integrating interval representations with fuzzy decision-making concepts, the model provides a flexible and adaptive mechanism for handling both quantitative uncertainty and subjective investor assessments in portfolio allocation decisions.

The proposed framework simultaneously incorporates several important investment dimensions, including expected return, portfolio risk, liquidity conditions, and diversification requirements across multiple investment periods, thereby supporting more sustainable and realistic long-term investment strategies. Since the original formulation constitutes an interval programming problem with fuzzy characteristics, a transformation process based on fuzzy decision-making theory is employed to convert the model into an equivalent crisp nonlinear programming formulation while preserving the uncertainty information embedded within the interval data structure. To solve the resulting optimization problem efficiently, this study adopts an enhanced Particle Swarm Optimization (PSO) algorithm equipped with adaptive inertia weights, time-varying acceleration coefficients, Gaussian local search mechanisms, and differential mutation strategies. These improvements are specifically designed to strengthen convergence stability, enhance solution accuracy, and reduce the risk of premature convergence to local optima.

The numerical simulation results demonstrate that the proposed framework is capable of effectively integrating investor preferences, expert judgments, and vague market assessments into the portfolio optimization process. The findings further indicate that the combination of interval analysis and fuzzy optimization techniques provides substantial methodological flexibility in dealing with uncertain financial environments and generates portfolio allocation strategies that are more adaptive, robust, and informative for multi-period investment decision-making.

REFERENCES

1. Alefeld, G., & Herzberger, J. (1983). *Introduction to interval computations*. Academic Press.
2. Alefeld, G., & Mayer, G. (2000). Interval analysis: Theory and applications. *Journal of Computational and Applied Mathematics, 121*(1–2), 421–464.
3. Bhattacharyya, R., Kar, S., & Dutta Majumder, D. (2011). Fuzzy mean-variance-skewness portfolio selection model with transaction costs. *Expert Systems with Applications, 38*(3), 1939–1947.
4. Buckley, J. J. (1989). Solving possibilistic linear programming problems. *Fuzzy Sets and Systems, 31*(3), 329–341.
5. Carlsson, C., Fullér, R., & Majlender, P. (2002). A possibilistic approach to selecting portfolios with highest utility score. *Fuzzy Sets and Systems, 131*(1), 13–21.
6. Chinneck, J. W., & Ramadan, K. (2000). Linear programming with interval coefficients. *Journal of the Operational Research Society, 51*(2), 209–220.
7. Du, W., & Li, B. (2008). Multi-strategy ensemble particle swarm optimization for global optimization. *Information Sciences, 178*(15), 3096–3109.
8. Dumas, B., & Luciano, E. (1991). An exact solution to a dynamic portfolio choice problem under transaction costs. *Journal of Finance, 46*(2), 577–595.
9. Fang, Y., Lai, K. K., & Wang, S. (2006). Portfolio rebalancing model with transaction costs based on fuzzy theory. *European Journal of Operational Research, 175*(2), 879–893.
10. Giove, S., Funari, S., & Nardelli, C. (2006). An interval portfolio selection problem based on regret criterion. *European Journal of Operational Research, 170*(1), 253–264.
11. Golmakani, H. R., & Fazel, M. (2011). Constrained portfolio selection using particle swarm optimization. *Expert Systems with Applications, 38*(7), 8327–8335.
12. He, Q., & Wang, L. (2007). A hybrid particle swarm optimization with a feasibility-based rule for constrained optimization. *Applied Mathematics and Computation, 186*(2), 1407–1422.
13. Ida, K. (2003). A multiobjective portfolio selection problem with interval coefficients. *Reliable Computing, 9*(1), 63–71.
14. Ida, K. (2004). Portfolio optimization with interval coefficients. *Applied Mathematics Letters, 17*(8), 913–918.
15. Inuiguchi, M., & Sakawa, M. (1995). Possibilistic linear programming: A brief review. *Fuzzy Sets and Systems, 74*(1), 3–13.
16. Inuiguchi, M., & Tanino, T. (2000). Portfolio selection under independent possibilistic information. *Fuzzy Sets and Systems, 115*(1), 83–92.
17. Ishibuchi, H., & Tanaka, H. (1990). Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research, 48*(2), 219–225.
18. Jana, D. K., Biswas, A., & Roy, T. K. (2009). Multi-objective portfolio selection using fuzzy programming. *International Journal of Mathematics in Operational Research, 1*(4), 435–457.
19. Jiang, C., Zhang, Z., & Han, X. (2008). Interval programming models for portfolio selection. *European Journal of Operational Research, 186*(1), 213–226.
20. Kapur, J. N. (1990). *Measures of information and their applications*. Wiley Eastern.

21. Kennedy, J., & Eberhart, R. (1995). Particle swarm optimization. In *Proceedings of IEEE International Conference on Neural Networks* (pp. 1942–1948).
22. Lai, K. K., Yu, L., Wang, S., & Huang, W. (2002). A note on portfolio selection with interval numbers. *International Journal of Information Technology & Decision Making, 1*(2), 243–255.
23. Li, X., & Xu, Z. (2007). Possibilistic mean–variance portfolio selection model with interval-valued returns. *Applied Mathematics and Computation, 185*(2), 1059–1067.
24. Li, D., Xu, Z., & Guo, S. (2010). Fuzzy mean–variance–skewness portfolio selection model. *Expert Systems with Applications, 37*(3), 2062–2070.
25. Liu, Y. J. (2011). Uncertain portfolio selection model with interval returns. *Applied Mathematics and Computation, 218*(3), 1047–1055.
26. Liu, Y. J., Zhang, W. G., & Xu, W. J. (2012). Multi-period portfolio selection model under fuzzy environment. *Economic Modelling, 29*(4), 113–119.
27. Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7*(1), 77–91.
28. Mitchell, J. (2001). Portfolio selection with interval returns. *Annals of Operations Research, 99*(1–4), 157–171.
29. Moore, R. E. (1966). *Interval analysis*. Prentice-Hall.
30. Mossin, J. (1968). Optimal multiperiod portfolio policies. *Journal of Business, 41*(2), 215–229.
31. Parra, M. A., Terol, A. B., & Uria, M. V. R. (2001). A fuzzy goal programming approach to portfolio selection. *European Journal of Operational Research, 133*(2), 287–297.
32. Pliska, S. R. (1997). *Introduction to mathematical finance: Discrete time models*. Blackwell.
33. Ramík, J., & Římánek, J. (1985). Inequality relation between fuzzy numbers and its use in fuzzy optimization. *Fuzzy Sets and Systems, 16*(2), 123–138.
34. Ramaswamy, S. (1998). Portfolio selection using fuzzy decision theory. *Journal of Banking & Finance, 22*(8), 1025–1041.
35. Ratnaweera, A., Halgamuge, S. K., & Watson, H. C. (2004). Self-organizing hierarchical particle swarm optimizer with time-varying acceleration coefficients. *IEEE Transactions on Evolutionary Computation, 8*(3), 240–255.
36. Sadjadi, S. J., Omrani, H., & Alinaghian, M. (2011). A multi-period fuzzy portfolio selection model. *Applied Soft Computing, 11*(3), 3093–3100.
37. Sun, J., Xu, W., & Feng, B. (2011). A global search strategy of quantum-behaved particle swarm optimization. *Applied Mathematics and Computation, 217*(14), 6545–6555.
38. Zadeh, L. A. (1965). Fuzzy sets. *Information and Control, 8*(3), 338–353.
39. Zhang, W. G., Liu, Y. J., & Xu, W. J. (2007). A possibilistic mean-semivariance model for portfolio selection. *European Journal of Operational Research, 176*(3), 1319–1330.
40. Zhang, W. G., Liu, Y. J., & Xu, W. J. (2009). Portfolio selection model based on fuzzy mean-variance. *Fuzzy Optimization and Decision Making, 8*(3), 255–272.
41. Zhang, W. G., Liu, Y. J., & Xu, W. J. (2010). A new fuzzy portfolio selection model. *Applied Mathematics and Computation, 216*(11), 3377–3387.
42. Zhang, W. G., Liu, Y. J., & Xu, W. J. (2012). Multi-period portfolio selection model with entropy. *Economic Modelling, 29*(4), 113–119.